Empirically Determining Point-Spread Functions

Michael Turmon
Machine Learning Group
Jet Propulsion Laboratory
Pasadena, CA 91109
mjt@aig.jpl.nasa.gov

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This note describes my ideas on determining the point-spread function from images in our data set. These are full-disk solar images taken in the CaII K spectral line at Sacramento Peak National Solar Observatory in New Mexico, and digitized at a resolution of roughly 2000 by 2000 pixels by R. Kariyappa.

1 Introduction

In our large database of solar images it is of interest to determine the point-spread function (PSF) of each image. This point-spread function is controlled mainly by "seeing" or atmospheric distortion, although in other situations the limited telescope aperture can play a role. In such situations, the PSF is frequently assumed to be rotationally symmetric. A Gaussian function

$$h(x, y) = e^{-r^2/2\sigma^2} \tag{1}$$

with $r = (x^2 + y^2)^{1/2}$ an implicit function of x and y is often used to fit the central part of the PSF, although the Lorentzian

$$h(x, y) = \frac{1}{1 + (r/r_0)^\beta} \tag{2}$$

[1]This work was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

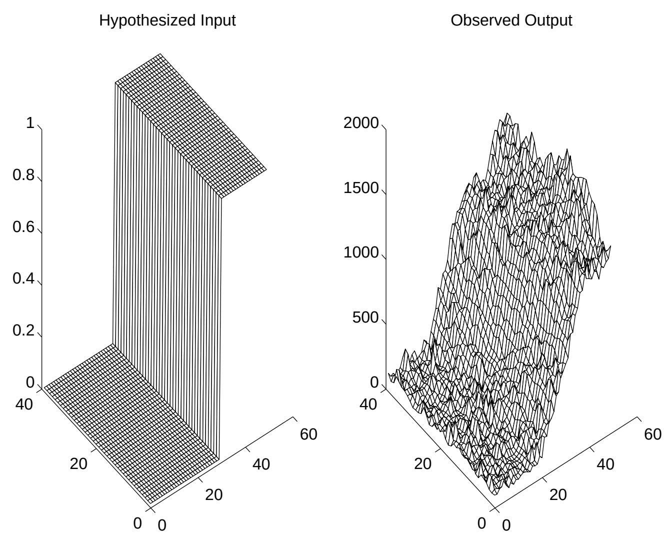
may capture the tail behavior more effectively. Sometimes a mixture of these two is used.

Since the PSF depends on seeing, it varies temporally. The Sacramento Peak K-line images we have are taken over about three minutes, and many believe that PSF variations over these time scales are not important. However, the PSF definitely varies from day to day. Following Worden [1], an indirect but simple way to assess the seeing is to form the histogram of the image after it has been adjusted for limb-darkening. Since the area of the sun is dominated by the so-called "quiet sun" component which are fluctuations about a mean level, the histogram is unimodal and its central part may be well-fit by a Gaussian. (There is a substantial tail from the bright network and plage components.) The width of the central Gaussian is monotonically related to seeing quality. The intuition is that if completely blurred, the image would be constant and the width of the Gaussian would be zero. Time series of such widths show variations of almost a factor of two (see p. 188 of [1]), so seeing variations appear to be significant.

To understand how the PSF can be determined from our image set, we need to briefly describe how they are distorted in their trip from the sun to the film. For now we neglect any problems in digitizing them. The signals originate in the chromosphere and we consider this their ideal form. They are attenuated by passing through a spatially varying shell of solat atmosphere. This shell is thicker at the edges than the center and the effect is called 'limb-darkening'. Then the signals pass through Earth's atmosphere and a telescope with corresponding point-spread functions; the overall PSF is the convolution of these. They then expose a photographic plate which responds nonlinearly to the incident light. The raw data for us consists of the digitized plates. These three effects (limb-darkening, blurring, film nonlinearity) must be compensated in reverse order. In particular, the appropriate place to estimate the PSF is on the images that have been corrected for film nonlinearity, but not for limb-darkening.

2 Finding PSF: Essentials

The idea of how to determine the PSF is simple. The basis is that we know the output of the blurring system, so to determine the PSF we have to postulate an input and then take the ratio of their (two-dimensional) Fourier transforms. One could assume that the un-degraded sun is a disc of unit height, and take this as the input and the whole available image as output. However, this is not true for at least three reasons: prominences make the true sun non-circular; the intensity across the sun is not uniform due to limb-darkening (for which exact expressions are not known) and plages; and our images are sometimes truncated on the edges. My proposed solution is to take as the output, an $M \times N$ section $\mathbf{y}$ of the edge of the sun which has no prominences or plages. For now let's assume $\mathbf{y}$ comes from the top part of the sun as seen on the film. If the size of $\mathbf{y}$ is small compared to the overall disc, we can take the input to be a unit step; that is, $\mathbf{x}$ is also $M \times N$ with the lower $M/2$ rows as all ones. Assuming the input has this form is the crucial element. See the figure below for an example input/output pair.



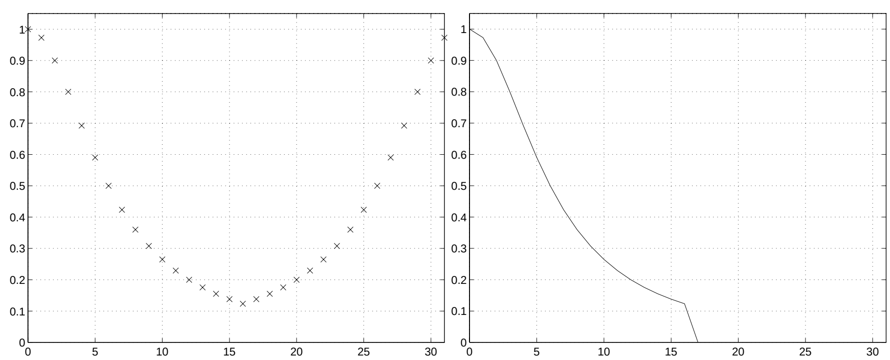
The two-dimensional Fourier transforms of $\mathbf{x}$ and $\mathbf{y}$ are $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ respectively, so the transform of the PSF is the pointwise quotient

$$\tilde{h}(k,l) = \frac{\tilde{y}(k,l)}{\tilde{x}(k,l)} \tag{3}$$

Unfortunately, as the input $x(m,n)$ is constant with respect to n, $\tilde{x}(k,l) = 0$ for all $l > 0$, and the transfer function is undefined there, so (3) fails. We put no energy into the system at these frequencies, so we can gather no information about the transfer function there.

To obtain the full transfer function, more information must be gathered. One way is to repeat the above process for other locations on the solar disc to fill in $\tilde{\mathbf{h}}$. This will work as long as the solar limb is largely free from plages and prominences, and is not truncated — these conditions are hard to fulfill simultaneously. It seems preferable to assume that the transfer function

$\mathbf{h}$ is radially symmetric (a function of r alone), which is true if the PSF is dominated by the atmospheric blurring component. In this case, the transfer function is also radially symmetric, and the unknown points can be filled in on this basis. That is, for integer $0 \le k \le M/2$, define $g(k) = \tilde{y}(k,0)/\tilde{x}(k,0)$, and interpolate linearly to define g for all real $0 \le k \le M/2$. (Since g, as the transfer function of a real-valued system, is symmetric about $M/2$, the highest frequency is $M/2$.) Define $g(k) = 0$ for $k > M/2$. If $|g(M/2)|$ differs significantly from zero, truncation is problematic — we return to this later. The left panel of the figure below shows the known values of such a transfer function. On the right is the corresponding linearly interpolated and truncated g function.



The transfer function is then filled in for all $|k| < M/2$ and $|l| < N/2$ according to $\tilde{h}(k,l) = g((k^2 + l^2)^{1/2})$. Note that just as a univariate transfer function is periodic in one variable, the bivariate transfer function $\tilde{\mathbf{h}}$ is periodic in two variables: $\tilde{h}(k,l) = \tilde{h}(k + \kappa_1 M, l + \kappa_2 N)$ for integer κ_1, κ_2. This observation allows us to extend the above definition of $\tilde{h}(k,l)$ to the whole (k,l) plane, and in particular to the domain $0 \le k < M$, $0 \le l < N$. The PSF is the inverse transform of $\tilde{\mathbf{h}}$.

3 Finding PSF: Details

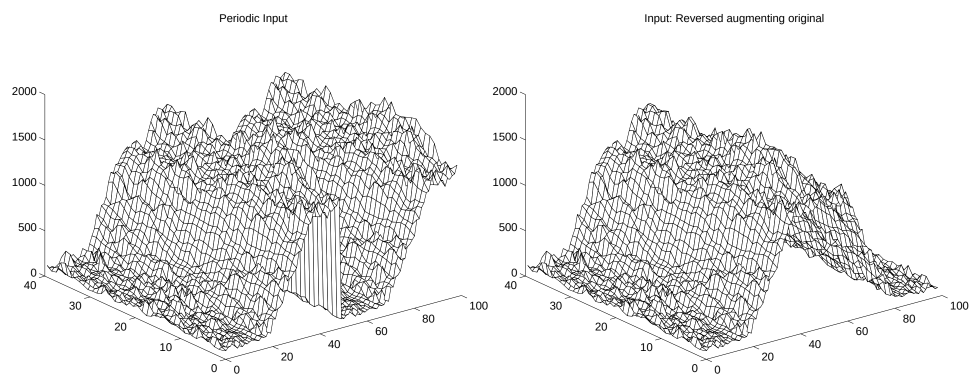
The details may be divided into finding the transfer function and getting from there to the PSF.

3.1 Image to transfer function

Say as before that an $M \times N$ chunk of the top edge of the solar image is used to get $\mathbf{y}$. Here M is the length of the response to the unit step, and N

essentially identical image rows will be used as $\mathbf{y}$. N is chosen as large as possible with the restrictions that the curvature of the sun is not significant, and no prominences or plages interfere with the imagined unit-step input.

Note that the 2-D FFT of $\mathbf{y}$ implicitly assumes that $\mathbf{y}$ is doubly periodic in M and N, so we do not want to take the unmodified image segment as $\mathbf{y}$ — this would in effect introduce another transition to zero at the edge of the window. See the figure below. To make the implicit transition smooth, we propose to augment both input and output image segments with duplicate, spatially-reversed versions of themselves. (This simple solution is available to us because we assume the PSF is radially symmetric; in particular, $h(m,n) = h(-m,n)$.) The hypothesized input then becomes a pulse or "boxcar" in one direction and contant in another. See the figure for the corresponding output.



Next, a time-saver in processing. The 2-D DFT of an image $\mathbf{x} = x(m,n)$ which is constant with respect to n is

$$\tilde{x}(k,l) = \frac{1}{M}\sum_{m=0}^{M-1} e^{-i\frac{2\pi km}{M}} x(m,n_0)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-i\frac{2\pi ln}{N}}\right] \tag{4}$$

for any n_0. The bracketed term is zero except for at $l = 0$, where it is unity. We conclude that the interesting part of the 2-D DFT of the input step (or pulse) is precisely the 1-D DFT of any of its columns. Further, note that we are only interested in the corresponding part of the 2-D DFT of the image segment $\mathbf{y}$, i.e. at $l = 0$, which becomes

$$\tilde{y}(k,0) = \frac{1}{M}\sum_{m=0}^{M-1} e^{-i\frac{2\pi km}{M}}\left[\frac{1}{N}\sum_{n=0}^{N-1} y(m,n)\right] \tag{5}$$

The bracketed term is now just the average across columns of $\mathbf{y}$, so the desired Fourier components can be obtained as the 1-D FFT of the *average* of $\mathbf{y}$; call this $\bar{y}$. The quantity g can therefore be found without a 2-D FFT.

The third detail is that the input $\mathbf{x}$ has been assumed to be a step, but this is not quite true due to limb-darkening, and I think this effect will be significant. As one moves from top to bottom along the solar disc the brightness, as hypothetically observed prior to atmospheric distortion, goes from zero to a large value (say 0.7 in arbitrary units) at the sun's edge, and then increases (quickly at first) to a maximum of 1.0 at disc center. This can be observed in a top-to-bottom cut through our digitized data.

I recommend the following work-around. The averaged response $\bar{y}$ to the step function starts out low and increases slowly, the rate of increase topping out at the "edge" of the step function, and thereafter $\bar{y}$ continues to increase, but more slowly. My view is that the first segment of this response can be regarded as being from the original step, but the latter part (after the first derivative of $\bar{y}$ peaks) should not be. The fix is to splice together two copies of the first segment of $\bar{y}$; the second part is reversed in time and in sign, and shifted up to fit with the first segment.

The output then becomes a pulse pieced together from four duplicated segments of the original averaged output $\bar{y}$. The input goes from zero to one and from one to zero just as the derivative of the output is maximized (first positive-going and second negative-going). The input and output are therefore mostly zero, with a narrow pulse. To obtain as much frequency resolution as possible, the input and output should be padded with zeros[2] to a length of about $M = 1024$. (The frequency resolution is one sample every $2\pi/M$ radians/sec.) The actual pulses should be made as narrow as possible. When I tested this procedure, I found it was easy to make the input one sample too wide or narrow for the output, with disastrous results for the quotient. It is easy to check the width of the input by plotting its DFT on a log scale with the DFT of the actual output. The several deep drops in the frequency representations should coincide.

3.2 Transfer function to PSF

The result of this process will be a length-1024 vector, akin to a one-dimensional transfer function, which we shall call f to distinguish it from g and $\tilde{\mathbf{h}}$. (It is f that is plotted in the left panel of the figure above.) It should

[2] Of course, the 'black' level in the image is not currently a numerical zero. It is simple to subtract this 'black' level from $\bar{y}$ so that its base level is indeed zero; this has no effect on the PSF.

be real, or very nearly so, and conjugate-symmetric ($f(k) = f(M-k)^*$). If f is not entirely real, check for a phase shift by plotting the phase angle of $f(k)$ versus k. If it is linear and makes an integer number of cycles from zero to 2π, this indicates a simple phase shift of the PSF. To undo this, just multiply $f(k)$ by $e^{-i\frac{2\pi k n_c}{M}}$ (or the conjugate of this) where n_c is the number of cycles made above. This sets the phase of the PSF to zero and should make the transfer function nearly real.

As mentioned above, the next step is to form g from f via interpolation and truncation. Ideally, f should go to zero at the highest frequency, but this may not be so. It is important not to allow g to drop roughly to zero as was done in the right panel above. (If this is done, the eventual PSF $\mathbf{h}$ will suffer from high-frequency 'ringing'.) Instead, I recommend the well-known technique from spectrum estimation known as 'windowing'. This means multiplying a transfer function (here g) by a (usually smooth) nonnegative 'window' $w(k)$. This is equivalent to convolving the PSF (in the spatial domain) with the inverse transform of w.[3]

I recommend still doing the truncation, but after that, multiplying by a w that is Gaussian with a width as great as possible. Hopefully g falls fast enough so that, without windowing,

$$\rho = |\frac{g(0)}{g(M/2)}| \leq 0.1 \tag{6}$$

If so, choose the width so that

$$\rho' = |\frac{w(0)g(0)}{w(M/2)g(M/2)}| \leq 0.05 \tag{7}$$

at least, and perhaps substantially smaller. This is a matter of judgement, because the window should not damp the central part of g too much. What you are doing is convolving the original $\mathbf{h}$ with the inverse DFT of a Gaussian, which is itself Gaussian with variance inversely related to that of w. So taking the inverse DFT of w will help to judge the amount of smoothing you are applying. It should be small in relation to the width of $\mathbf{h}$, of course. (There is a substantial knowledge about these windowing functions, e.g. [2], and we can do a better job than indicated above. However, I believe that error due to inappropriate windowing is dominated by error in the hypothesized input.)

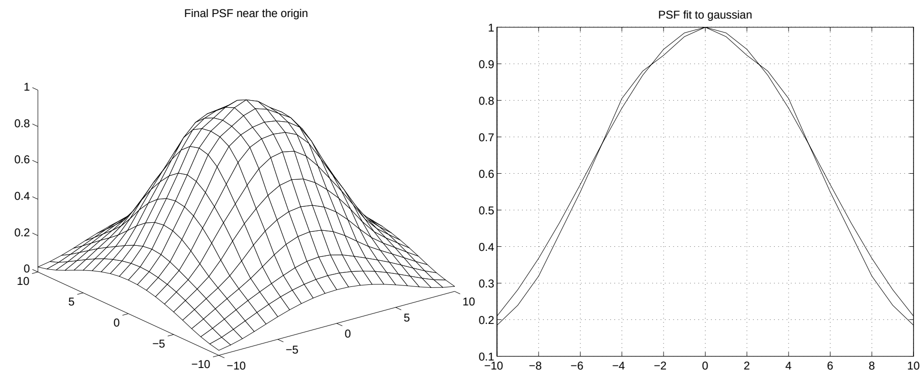
[3] One way to think of the previously-described process of truncating g to get f is that f was multiplied by a pulse-shaped window that was zero beyond position $M/2$. The inverse DFT of the pulse is a $sin(m)/m$-type function, which is oscillatory and causes the ringing mentioned above.

Up to now, we have worked in one dimension, except for the original step of removing a perhaps 64×64 piece from the image. From this piece, we have obtained a 'radial transfer function' f, which is of length perhaps 1024, and the truncated, windowed function g, also of length about 1024. Now this must be interpolated to generate a large square transfer function $\tilde{\mathbf{h}}$ that is $M \times M$, and this is the real limit to the computation.

3.3 Sanity checks

Then $\mathbf{h}$, the inverse 2-dimensional DFT of $\tilde{\mathbf{h}}$, can be found. Just as $\tilde{\mathbf{h}}$ was a bump centered about $(0, 0)$, so should $\mathbf{h}$ be. The point is that we expect $h(0, 0)$ to be significant, and also $h(1, 0)$, and also $h(-1, 0)$ and its periodic twin $h(M - 1, 0)$ — components of the bump are in all four corners of $\mathbf{h}$. However they should be confined fairly tightly to those corners; I expect little if any energy beyond about 50 units from the origin if indeed $M = 1024$. Of course the PSF should be purely real, at least to numerical accuracy. Reality is assured from the symmetry of $\tilde{\mathbf{h}}$ ($\tilde{h}(k, l) = \tilde{h}(-k, l)^* = \tilde{h}(k, -l)^*$) which is obtained as the "circularized" g. A nontrivial accuracy check is that any blurring-dominated PSF has $(\forall m)(\forall n)h(m, n) \geq 0$, although this may be violated slightly. Finally, as mentioned at the outset, the width of the main peak in the histogram of overall image intensity is related to seeing quality, which is in turn inversely related to PSF width. This could function as a check on this method if several images are analyzed.

Below is a sample calculation based on some of the data seen above, which is from a Sacramento Peak image from January, 1992. The left panel shows the computed PSF near the origin; it must be radially symmetric by construction but it is also nonnegative and varies in a reasonable way. The right panel shows the (rather good) fit of the central part to a Gaussian. The PSF FWHM ("full-width half-max") is about a dozen pixels. Since the solar disc in the image comprises 2000 pixels, the FWHM is on the order of 20 arcseconds.



References

[1] J. Worden. *A Three Component Proxy Model for the Solar Far Ultraviolet Irradiance*. PhD thesis, Univ. Colorado, Boulder, CO, 1996.

[2] D.J. Thomson. Spectrum estimation and harmonic analysis. *Proceedings of the IEEE*, 70(9):1055–1096, September 1982.